MATH 211 SPRING 2014 TEST I VERSION A

NAME: KEY SECTION:

(1) True/False Questions (2 points each): Here A, B etc. denote sets while $x, y, \cdots$ denote elements, and $\mathcal{P}(A)$ denotes the power set of a set A

T F $\{\phi\} \subset \{\phi, \{\{\phi\}\}\}$

T F $\mathcal{P}(\phi) = \{\{\phi\}\}$

T F $\pi \in \{\mathbb{R}, \mathbb{N}, \mathbb{Q}\}$, where π is the well-known number

T F Disjunctive Syllogism uses the tautology $((p \vee q) \wedge \neg p) \rightarrow q$

T F The proposition $(\neg p \rightarrow q) \rightarrow (q \rightarrow p)$ is not a contingency

T F Let $\{a_n\}$ be a geometric progression with first term $a = 1$ and common ratio $r = 2$. Then $\sum_{k=9}^{19} a_n = 2^{20} - 2^{10}$.

T F A and B are two sets such that $card(A) = card(B) = 3$, then $card(\mathcal{P}(A \times B)) = 2^3 \times 2^3$

T F For any $f : A \rightarrow B$, S any subset of A we have $f^{-1}(f(S)) \subset S$

T F There exist two sets A and B such that A is both an element of B as well as a subset of B at the same time

T F $(A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (A \cap B) = A \cup B$

T F existential instantiation states that if $\exists x P(x)$ is true, then you can prove that $P(c)$ is true for an arbitrarily selected member in the domain.

T F Let $f : A \rightarrow B$ be a one-to-one function, let S and T be two two subsets of A, then $f(S \cap T) = f(S) \cap f(T)$

T F A universal quantifier has a higher precedence over the logical operator $\wedge$

T F The inverse of the proposition "you need to buy gasoline whenever you drive more than 400 kilometers" is "if you need to buy gasoline, then you would have driven more than 400 kilometers"

(2) (6 pt.) Find a bijective function f from $\mathbb{N}$ to $\mathbb{Z}$. Make sure you prove that your function is both one to one and onto.

$\mathbb{N}$: 0 1 2 3 4 5 6 ....

$\mathbb{Z}$: 0 -1 1 -2 2 -3 3

using this alignment, define $f(n) = \begin{cases} \frac{n}{2}; & \text{if } n \text{ is even} \\ -\frac{(n+1)}{2} & \text{if } n \text{ is odd} \end{cases}$

It is 1-1: suppose $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$

case i) their common value is > 0. then they must be both even, because no odd value in $\mathbb{N}$ is sent to a positive value, so $\frac{n_1}{2} = \frac{n_2}{2} \Rightarrow n_1 = n_2$.

Case ii) if the common value is < 0, then both n_1 & n_2 must be odd. so $-\frac{(n_1+1)}{2} = -\frac{(n_2+1)}{2} \Rightarrow n_1 + 1 = n_2 + 1 \Rightarrow n_1 = n_2$.

It is Onto: Let $z \in \mathbb{Z}$. If $z > 0$, then $2z$ is preimage: $f(2z) = \frac{2z}{2} = z$.

If $z < 0$, then $n = 1 - 2z$ is a preimage: $f(1-2z) = -\frac{(1-2z+1)}{2} = z$

so $\forall z$, z has a pre-image, therefore f is onto.

(3) (6 pt.) Consider the function $f : \mathbb{Z}^+ \to \mathbb{Z}$ defined by $f(a) = 3a^2 + 1$. Is f one-to-one? Is f onto?

one to one: Let a_1 & $a_2 \in \mathbb{Z}^+$ be s.t $f(a_1) = f(a_2)$, so $3a_1^2 + 1 = 3a_2^2 + 1 \Rightarrow 3a_1^2 = 3a_2^2 \Rightarrow a_1^2 = a_2^2$. Hence $a_1 = a_2$ because we know that they are both positive. so f is 1-1.

<u>f is NOT onto</u>: Let $z < 0$ & suppose z has a preimage, then $3a^2 + 1 = z$. But $3a^2 + 1 > 0$, hence $z > 0$. This contradicts the assumption that $z < 0$, so z has no pre-image and therefore f is not onto.

(4) Use logical equivalences to prove the following statements, naming each logical equivalence you use.

(a) (6 pt.) Show that the two propositions $(p \vee q) \to (p \wedge q)$ and $q \leftrightarrow p$ are logically equivalent.

$(p \vee q) \to (p \wedge q)$

$\equiv \neg(p \vee q) \vee (p \wedge q)$ using $p \to q \equiv \neg p \vee q$

$\equiv (\neg p \wedge \neg q) \vee (p \wedge q)$ de Morgan's law

$\equiv (\neg p \vee p) \wedge (\neg p \vee q) \wedge (\neg q \vee p) \wedge (\neg q \vee q)$ distributive law

$\equiv (\neg p \vee q) \wedge (\neg q \vee p) \wedge T$ negation law, commutative & assoc.

$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$ identity law $p \wedge T \equiv p$

$\equiv (p \to q) \wedge (q \to p)$ using $p \to q \equiv \neg p \vee q$

$\equiv (p \leftrightarrow q)$ by definition of bi conditional

(b) (6 pt.) Prove $\neg(p \to q) \to p$ is a tautology.

$\neg(p \to q) \to p$

$\equiv \neg(\neg(p \to q)) \vee p$ using $p \to q \equiv \neg p \vee q$

$\equiv (p \to q) \vee p$ double negation

$\equiv (\neg p \vee q) \vee p$ using $p \to q \equiv \neg p \vee q$

$\equiv \neg p \vee (q \vee p)$ associative law

$\equiv \neg p \vee (p \vee q)$ commutative law

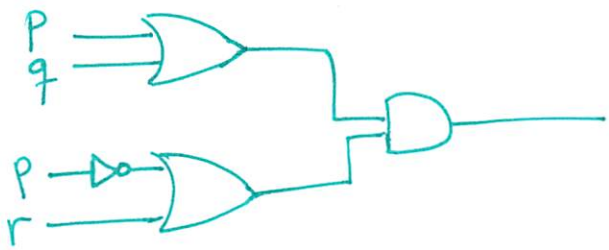
$\equiv (\neg p \vee p) \vee q$ associative law

$\equiv T \vee q$ negation law

$\equiv T$ domination

Hence the above proposition <u>is</u> a tautology.

(5) (6 pt.) Draw a combinatorial circuit that yields the output $(p \vee q) \wedge (\neg p \vee r)$. Use the smallest number of logical gates possible for this problem. Hint: you may simplify the given proposition using logical equivalences first.



Remark: Note that $(p \vee q) \wedge (\neg p \vee r) \longrightarrow (q \vee r)$ is the resolution tautology. However $(p \vee q) \wedge (\neg p \vee r) \longleftrightarrow (q \vee r)$ is not a tautology [Consider, e.g. $p = T$, $q = T$, $r = F$] so the circuit cannot by reduced to

(6) (6 pt.) Let A and B be two sets. Show that if $A \subset B$ then $A - B \subset B - A$. Which method of proof did you use?

We will use a direct proof: suppose that $A \subset B$ so if $x \in A$ then $x \in B$. We claim that $A - B = \emptyset$. If not, then $\exists x \in A - B$. Hence $x \in A$ and $x \notin B$. So $x \in B$ and $x \notin B$, a contradiction.

Since the empty set is a subset of any set, this proves that $A - B \subset B - A$.

(7) (6 pt.) Prove that $\sqrt[4]{2}$ is irrational.

[imitate the proof that $\sqrt{2}$ is irrational].

suppose $\sqrt[4]{2}$ is rational, then there exist two integers a and b, with $b \neq 0$ such that $\sqrt[4]{2} = \frac{a}{b}$. Raising both sides to the fourth power we get $2 = \frac{a^4}{b^4}$

we can further assume that a and b have no common factor, because if they do, we simply reduce them all, so that the resulting two numbers will have no common factors

Hence $a^4 = 2b^4$. so a^4 is even, so a^2 is even (proven) similarly a is then even. so that $a = 2K$ for some integer K. Then $a^4 = 16K^4$ and $16K^4 = 2b^4$. Hence $b^4 = 8K^4$ is also even. Like before, this implies that b is even. We have thus found that a and b have 2 as a common factor. contradiction. hence $\sqrt[4]{2}$ must be irrational.

(8) (6 pt.) Prove or disprove that if x is any irrational number then so is $\sqrt{x}$.

By contraposition, assume that $\sqrt{x}$ is not irrational, then it is rational and there exist two integers a & b, $b \neq 0$ such that $\sqrt{x} = \frac{a}{b}$. Therefore,

$x = \sqrt{x} \cdot \sqrt{x} = \frac{a}{b} \cdot \frac{a}{b} = \frac{a^2}{b^2}$, which is also rational. Then x is not irrational and this finishes the proof.

(9) In the following let the universe of discourse for x be all students, and the universe of discourse for y be all European countries. Consider the premises: $S(x)$: "x is a student in my class"; $V(x, y)$ is "student x visited country y". Translate the following statements into logical expressions:

(a) (6 pt.) There is a student in my class who never visited any European country:

$\exists x \forall y (\neg V(x, y))$

(b) (6 pt.) There is a European country that was visited by more than one student in my class:

$\exists y \exists x_1 \exists x_2 (x_1 \neq x_2 \wedge S(x_1) \wedge S(x_2) \wedge V(x_1, y) \wedge V(x_2, y))$

(10) Using the predicates of the previous problem translate the following to English:

(a) (6 pt.) $\exists x \exists y_1 (\neg V(x, y_1) \wedge \forall y_2 (y_1 \neq y_2 \rightarrow V(x, y_2)))$

There is a student who visited all but one European Country

(b) (6 pt.) $\exists x_1 \exists x_2 (x_1 \neq x_2 \wedge \forall y (\neg V(x_1, y) \leftrightarrow V(x_2, y)))$

There are two students who between them, have visited all European Countries but they never visited the same Country.

MATH 211 SPRING 2014 TEST I VERSION B

NAME: SECTION:

(1) True/False Questions (2 points each): Here A, B etc. denote sets while $x, y, \cdots$ denote elements, and $\mathcal{P}(A)$ denotes the power set of a set A

T F For any $f : A \to B$, S any subset of A we have $S \subset f^{-1}(f(S))$

T F Let $\{a_n\}$ be a geometric progression with first term $a = 1$ and common ratio $r = 2$. Then $\sum_{k=9}^{19} a_n = 2^{20} - 2^9$.

T F $(A \cap \bar{B}) \cup (\bar{A} \cap B) = A$

T F Let $f : A \to B$ be a function, let S and T be two two subsets of A, then $f(S) \cap f(T) \subset f(S \cap T)$

T F There exist two sets A and B such that A is both an element of B as well as a subset of B at the same time

T F The inverse of the proposition "you need to buy gasoline whenever you drive more than 400 kilometers" is "if you need to buy gasoline, then you would have driven more than 400 kilometers"

T F $\{\phi\} \subset \{\phi, \{\{\phi\}\}\}$

T F If $A = \{a, b\}$, $B = \{c, d\}$, then $(\{a\}, \{c\})$ is an element of $\mathcal{P}(A \times B)$

T F A universal quantifier has a higher precedence over the logical operator $\wedge$

T F $\mathcal{P}(\phi) = \{\{\phi\}\}$

T F $\pi \notin \{\mathbb{R}, \mathbb{N}, \mathbb{Z}\}$, where π is the well-known number

T F The proposition $(\neg p \to q) \to (q \to p)$ is a contingency

T F Universal generalization states that $\forall x P(x)$ is true, provided that you can prove that $P(c)$ is true for a specific member in the domain.

T F Modus tollens uses the tautology $p \wedge (p \to p) \to q$

TEST VERSION (B)

(9) In the following let the universe of discourse for x be all African countries, and the universe of discourse for y be all students. Consider the premises: $S(y)$: "y is a student in my class"; $V(x, y)$ is "country x was visited by student y". Translate the following statements into logical expressions:

(a) (6 pt.) There is a student in my class who never visited any African country:

$\exists y \forall x (\neg V(x, y) \wedge S(y))$

(b) (6 pt.) There is an African country that was visited by more than one student in my class:

$\exists x \exists y_1 \exists y_2 (y_1 \neq y_2 \wedge S(y_1) \wedge S(y_2) \wedge V(x, y_1) \wedge V(x, y_2))$

(10) Using the predicates of the previous problem translate the following to English:

(a) (6 pt.) $\exists y \exists x_1 (\neg V(x_1, y) \wedge \forall x_2 (x_1 \neq x_2 \rightarrow V(x_2, y)))$

There is a student who visited all African Countries except one

(b) (6 pt.) $\exists y_1 \exists y_2 (y_1 \neq y_2 \wedge \forall x (\neg V(x, y_1) \leftrightarrow V(x, y_2)))$

There are two students who have never visited the same African country, but between them, they visited all African Countries